

# Von Neumann Uniqueness Theorem doesn't Hold in Hyperbolic Quantum Mechanics

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It is shown that Von Neumann Uniqueness Theorem doesn't hold in Hyperbolic Quantum Mechanics.

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**KEY WORDS:** Hyperbolic Hilbert space; quantum mechanics; canonical commutation relations; unitary equivalence; hyperbolic Fourier transform.

## 1. INTRODUCTION

Following Adler's (1995) formalization of Feynman's basic observations (Feynman and Hibbs, 1965) concerning quantum probabilities, let us recall that a key feature of quantum probabilities consists in that they don't obey the usual *formula of probabilities' composition*:

$$P_{ca} = \sum_b P_{cb} \cdot P_{ba} \quad (1.1)$$

but a *formula for probabilities amplitudes's composition*:

$$\Phi_{ca} = \sum_b \Phi_{cb} \cdot \Phi_{ba} \quad (1.2)$$

where the probabilities amplitudes  $\Phi$ 's take value on a finite dimensional algebra  $A$  over  $\mathbb{R}$  (Shafarevich, 1997) on which a *modulus function*  $N : A \mapsto \mathbb{R}$  is defined such that:

$$P_{ba} = N^2(\Phi_{ba}) \quad (1.3)$$

$$P_{cb} = N^2(\Phi_{cb}) \quad (1.4)$$

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$$\Phi_{ca} = N^2(\Phi_{ba}) \tag{1.5}$$

where both the algebra  $A$  and the modulus function has to be determined imposing reasonable physical and mathematical constraints.

From a mathematical side it is natural to require that  $N$  is a norm over  $A$ .

From a physical side the imposition of the Correspondence Principle requires that, in the absence of quantum interference effects, probability amplitude superposition (i.e. Eq. (1.2)) should reduce to probability superposition (i.e. Eq. (1.1)). This leads (cfr. Adler, 1995 for details) to the condition that the norm  $N$  has to be multiplicative.

One has that:

**Theorem 1.1.** (Albert’s Theorem)

HP:

A finite dimensional algebra with unit over  $\mathbb{R}$

$N$  multiplicative norm over  $A$

TH:

$$A \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$$

where  $\mathbb{H}$  is the (noncommutative) algebra of Hamilton’s *quaternions* and  $\mathbb{O}$  is the (noncommutative and nonassociative) algebra of Cayley’s *octonions* whose definition we briefly recall.

The generic element of an  $(n + 1)$ -dimensional algebra with unit  $A$  may be expressed as:

$$\Phi = \sum_{i=0}^n r_i e_i \tag{1.6}$$

where  $e_0 = 1, \dots, e_n$  are the basis elements of the algebra obeying multiplication law:

$$e_i \cdot e_j = \sum_{k=0}^n f_{ijk} e_k \quad i, j = 0, \dots, n \tag{1.7}$$

with the real-valued structure constant  $f_{ijk}$ ’s obeying the following constraints:

$$f_{0ij} = \delta_{ij} \quad i, j = 0, \dots, n \tag{1.8}$$

$$f_{i0j} = \delta_{ij} \quad i, j = 0, \dots, n \tag{1.9}$$

that may be immediately derived imposing that:

$$e_i \cdot e_0 = e_0 \cdot e_i = e_0 \quad i = 0, \dots, n \tag{1.10}$$

The algebra of quaternions  $\mathbb{H}$  corresponds to the case  $n = 3$  and:

$$e_i \cdot e_j = -\delta_{ij} + \sum_{k=1}^3 \epsilon_{ijk} e_k \tag{1.11}$$

where  $\epsilon_{ijk}$  is the Levi Civita's tensor, i.e. the totally antisymmetric tensor with  $\epsilon_{123} = 1$ .

A multiplicative norm on  $\mathbb{H}$  is given by:

$$N \left( \sum_{i=0}^3 r_i e_i \right) = \sqrt{\sum_{i=0}^3 r_i^2} \tag{1.12}$$

The algebra of octonions  $\mathbb{O}$  corresponds to the case  $n = 7$  and:

$$e_i \cdot e_j = -\delta_{ij} + \sum_{k=1}^3 \gamma_{ijk} e_k \tag{1.13}$$

where  $\gamma_{ijk}$  is the totally antisymmetric tensor such that:

$$\gamma_{123} = \gamma_{246} = \gamma_{435} = \gamma_{367} = \gamma_{651} = \gamma_{572} = \gamma_{714} = 1 \tag{1.14}$$

A multiplicative norm on  $\mathbb{O}$  is given by:

$$N \left( \sum_{i=0}^7 r_i e_i \right) := \sqrt{\sum_{i=0}^7 r_i^2} \tag{1.15}$$

Let us observe, anyway, that from a physical viewpoint it seems rather natural to require that:

$$\Phi_{ab} \cdot \Phi_{bc} = \Phi_{bc} \cdot \Phi_{ab} \tag{1.16}$$

that leads to the constraint that  $A$  has to be commutative; so, by Albert Theorem,  $A \in \{\mathbb{R}, \mathbb{C}\}$ .

Since  $\mathbb{R}$  is too strict to allow all the superpositions of states observed experimentally, it follows that Nature has chosen  $A = \mathbb{C}$ .

Let us now observe that, with the exception of the non-associative octonions' algebra  $\mathbb{O}$ , the algebras allowed by Albert's Theorem are finite-dimensional real Clifford algebras (Gurlebeck and Sprossig, 1997):

$$\mathbb{R} = Cl_{0,0} \tag{1.17}$$

$$\mathbb{C} = Cl_{0,1} \tag{1.18}$$

$$\mathbb{H} = Cl_{0,2} \tag{1.19}$$

To understand better the structural properties of the choice of  $Cl_{0,1}$  made by Nature, it may be interesting to investigate how the mathematical structure of

Quantum Mechanics is modified by the ansatz:

$$Cl_{0,1} \mapsto Cl_{1,0} \quad (1.20)$$

corresponding, in terms of the structure constants, to the ansatz:

$$f_{111} = 0 \quad f_{110} = 1 \mapsto f_{111} = 0 \quad f_{110} = -1 \quad (1.21)$$

Since, as we have shown in Khrennikov and Segre (2005), there are many reasons to call  $\mathbb{G} := Cl_{1,0}$  the *hyperbolic algebra*, we will denote the  $Cl_{1,0}$ -Quantum Mechanics as Hyperbolic Quantum Mechanics.

Such a mathematical theory emerged in the research of one of the authors (Khrennikov, 2003a, 2003b, 2004) and of other scientists (see Kocik, 1999; Rochon and Tremblay, 2004; Ulrych, 2005a, 2005b, 2005c; Xuegang, 2000; Zheng and Xuegang, 2004 and references therein).

Let us observe, first of all, that the modulus function:

$$N \left( \sum_{i=0}^1 r_i e_i \right) := \sqrt{\sum_{i=0}^1 r_i^2} \quad (1.22)$$

is a norm, though not multiplicative.

**Remark 1.1.** We would like to stress, from the physical point of view, that we are in no way claiming that Quantum Mechanics, as a physical theory, is wrong or has to be modified.

We are simply analyzing an alternative mathematical theory whose structure could allow to get some insight of the  $Cl_{0,1}$ -choice made by Nature.

## 2. THE HYPERBOLIC ALGEBRA

Let us define the hyperbolic algebra as the ring  $\mathbb{G}$  of numbers of the form  $x + jy$ , where  $x, y \in \mathbb{R}$  while  $j$ , called the hyperbolic imaginary unit, is such that  $j^2 = +1$ .

The elements of such an algebra has been called in the mathematical literature with different names (cfr. Janczewicz, 1996 and references therein): *hyperbolic numbers*, *double numbers*, *split complex numbers*, *perplex numbers*, and *duplex numbers*.

We will call them *hyperbolic numbers* and we will refer to  $j$  as to the *hyperbolic imaginary unit*.

The complex field  $\mathbb{C}$  and the hyperbolic ring  $\mathbb{G}$  are the two bidimensional Clifford algebras (Khrennikov and Segre, 2005):

$$\mathbb{C} = Cl_{0,1} \quad (2.1)$$

$$\mathbb{G} = Cl_{1,0} \quad (2.2)$$

Given  $z = x + jy \in \mathbb{G}$ :

*Definition 2.1.* (Conjugate of  $z$ )

$$\bar{z} := x - jy$$

*Definition 2.2.* (Norm of  $z$ )

$$\|z\| := \sqrt{x^2 + y^2}$$

*Definition 2.3.* (Light Cone of  $z$ )

$$V_z := \{z' = x' + jy' \in \mathbb{G} : y' = \pm(x' - x) + y\}$$

One has that:

**Theorem 2.2.**

- $\mathbb{G}$  is a commutative ring
- $\mathbb{G}$  is not a field

**Proof:**

1. by definition the addition and the multiplication in  $\mathbb{G}$  are commutative and associative, the multiplication is distributive with respect to the addition, there exists a null element 0 with respect to addition, there exists an identity element 1 with respect to multiplication and every element of  $\mathbb{G}$  has an additive inverse
2. given  $z = x + jy \in \mathbb{G}$  one has formally that:

$$\frac{1}{z} = \frac{1}{x + jy} = \frac{x}{x^2 - y^2} - j \frac{y}{x^2 - y^2} \tag{2.3}$$

and hence:

$$\exists z^{-1} \Leftrightarrow z \notin V_0 \tag{2.4}$$

So not every nonzero element of  $\mathbb{G}$  has a multiplicative inverse and hence  $\mathbb{G}$  is not a field

□

One has clearly that:

**Proposition 2.1.**

$$\|z\|^2 \neq z\bar{z} \quad \forall z \in \mathbb{G} - \mathbb{R}$$

**Proof:** Given  $z = x + jy \in \mathbb{G} - \mathbb{R}$  one has that

$$\|z\|^2 = x^2 + y^2 \neq z\bar{z} = x^2 - y^2 \quad (2.5)$$

□

### 3. HYPERBOLIC HILBERT SPACES

Let us start introducing the following:

*Definition 3.1.* (Hyperbolic Linear Space) a triple  $(V, +, \cdot)$  where  $V$  is a set while  $+: V \times V \mapsto V$  and  $\cdot: \mathbb{G} \times V \mapsto V$  are such that:

$$\begin{aligned} u + v &= v + u \quad \forall u, v \in V \\ (u + v) + w &= u + (v + w) \quad \forall u, v, w \in V \\ \exists 0 \in V : u + 0 &= u \quad \forall u \in V \\ \lambda(u + v) &= \lambda u + \lambda v \quad \forall u, v \in V, \forall \lambda \in \mathbb{G} \\ (\lambda + \mu)u &= \lambda u + \mu u \quad \forall u \in V, \forall \lambda, \mu \in \mathbb{G} \\ 1u &= u \quad \forall u \in V \end{aligned}$$

We can then introduce the following:

*Definition 3.2.* (Hyperbolic Inner Product Space) a quadruple  $(V, +, \cdot, (\cdot, \cdot))$  such that:

- $(V, +, \cdot)$  is an hyperbolic linear space
- $(\cdot, \cdot): V \times V \mapsto \mathbb{G}$  is such that:

$$\begin{aligned} (u, v + w) &= (u, v) + (u, w) \quad \forall u, v, w \in V \\ (u, \lambda v) &= \lambda(u, v) \quad \forall u, v \in V, \forall \lambda \in \mathbb{G} \\ (u, v) &= \overline{(v, u)} \quad \forall u, v \in V \end{aligned}$$

*Example 3.1.* Let  $\mathbb{G}^n$  denote the set of all n-ples of hyperbolic numbers; given  $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{G}^n$  define:

$$(x, y) := \sum_{i=1}^n \bar{x}_i y_i \quad (3.1)$$

$(\mathbb{G}^n, (\cdot, \cdot))$  is then an hyperbolic inner product space

Given an hyperbolic inner-product space  $(V, +, \cdot, (\cdot, \cdot))$  and a linear operator  $U : V \mapsto V$ :

*Definition 3.3.* (U is Unitary)

$$(Ux, Uy) = (x, y) \quad \forall x, y \in V$$

*Definition 3.4.* (Hyperbolic Normed Linear Space) a couple  $(V, \|\cdot\|)$  where:

- $V$  is an hyperbolic linear space
- $\|\cdot\|$  is a map  $\|\cdot\| : \mathbb{G} \mapsto \mathbb{R}$  such that:

$$\|v\| \geq 0 \quad \forall v \in V$$

$$\|v\| = 0 \quad \Leftrightarrow \quad v = 0$$

$$\exists c \in \mathbb{R}_+ : (\|\alpha v\| \leq c\|\alpha\|\|v\| \quad \forall v \in V, \forall \alpha \in \mathbb{G})$$

$$\|u + v\| \leq \|u\| + \|v\| \quad \forall u, v \in V$$

Given two hyperbolic normed linear spaces  $(V_1, \|\cdot\|_1)$  and  $(V_2, \|\cdot\|_2)$  and a linear operator  $T : V_1 \mapsto V_2$ :

*Definition 3.5.* (T is Bounded)

$$\|T\| := \sup_{\|v\|_1=1} \|Tv\|_2 \in \mathbb{R}$$

*Definition 3.6.* (Hyperbolic Banach Space) an hyperbolic normed linear space  $(V, \|\cdot\|)$  which is complete as a metric space in the induced metric  $d(u, v) := \|u - v\|$ .

*Definition 3.7.* (Hyperbolic Hilbert Space) a triple  $(V, (\cdot, \cdot), \|\cdot\|)$  such that:

- $(V, (\cdot, \cdot))$  is an hyperbolic inner-product space
- $(V, \|\cdot\|)$  is an hyperbolic Banach space

$$\exists c \in \mathbb{R}_+ : (\|(u, v)\| \leq c\|u\|\|v\| \quad \forall u, v \in V)$$

**Remark 3.1.** Let us observe that all the introduced notions of the form “hyperbolic x,” with  $x =$  linear space, normed linear space, Banach space, Hilbert space, has not to be intended as particular cases of the respective notion  $x$ :

Since  $\mathbb{G}$  is not a field, an hyperbolic linear space is not a linear space but only a module over the ring  $\mathbb{G}$  and so on.

*Example 3.2.* Given the hyperbolic inner product space  $(\mathbb{G}^n, (\cdot, \cdot))$  let us introduce the hyperbolic Banach space  $(\mathbb{G}^n, \|\cdot\|)$  defined as:

$$\|(x_1 + jy_1, \dots, x_n + jy_n)\| := \sqrt{\sum_{i=1}^n (x_i^2 + y_i^2)} \tag{3.2}$$

Given  $x = (a_1 + jb_1, \dots, a_n + jb_n) \in \mathbb{G}^n$  and  $\alpha = c + jd \in \mathbb{G}$  one has that:

$$\|\alpha x\| = \sqrt{\sum_{i=1}^n (c^2 a_i^2 + c^2 b_i^2 + d^2 a_i^2 + d^2 b_i^2 + 4a_i b_i c d)} \tag{3.3}$$

$$\|\alpha\| \|x\| = \sqrt{\sum_{i=1}^n (c^2 a_i^2 + c^2 b_i^2 + d^2 a_i^2 + d^2 b_i^2)} \tag{3.4}$$

Since:

$$2a_i b_i c d \leq c^2 a_i^2 + d^2 b_i^2 \tag{3.5}$$

$$2a_i b_i c d \leq c^2 b_i^2 + d^2 a_i^2 \tag{3.6}$$

it follows that:

$$\|\alpha x\| \leq \sqrt{2} \|\alpha\| \|x\| \tag{3.7}$$

Furthermore, given  $x = (a_1 + jb_1, \dots, a_n + jb_n), y = (c_1 + jd_1, \dots, c_n + jd_n) \in \mathbb{G}^n$  one has that:

$$\|(x, y)\| = \sqrt{\sum_{i=1}^n (a_i^2 c_i^2 + a_i^2 d_i^2 + b_i^2 c_i^2 + b_i^2 d_i^2 - 4a_i b_i c_i d_i)} \tag{3.8}$$

$$\|x\| \|y\| = \sqrt{\sum_{i=1}^n (a_i^2 c_i^2 + a_i^2 d_i^2 + b_i^2 c_i^2 + b_i^2 d_i^2)} \tag{3.9}$$

Since:

$$-2a_i b_i c_i d_i \leq a_i^2 c_i^2 + b_i^2 d_i^2 \tag{3.10}$$

$$-2a_i b_i c_i d_i \leq a_i^2 d_i^2 + b_i^2 c_i^2 \tag{3.11}$$

it follows that:

$$\|(x, y)\| \leq \sqrt{2} \|x\| \|y\| \tag{3.12}$$



$(\mathbb{G}^n, (\cdot, \cdot), \|\cdot\|)$  is then an hyperbolic Hilbert space.

*Example 3.3.* Define  $L^2(\mathbb{R}, \mathbb{G})$  to be the set of hyperbolic valued measurable functions on  $\mathbb{R}$  that satisfy  $\int_{-\infty}^{+\infty} dx \|f(x)\|^2 < +\infty$ . Let us introduce:

$$(f, g) := \int_{-\infty}^{+\infty} dx \bar{f}(x)g(x) \tag{3.13}$$

and:

$$\|\psi\| := \sqrt{\int_{-\infty}^{+\infty} dx \|\psi(x)\|^2} \tag{3.14}$$

One has that:

$$\begin{aligned} \|z\psi\| &= \sqrt{\int_{-\infty}^{+\infty} dx \|z\psi(x)\|^2} \leq \sqrt{2} \sqrt{\int_{-\infty}^{+\infty} dx \|z\|^2 \|\psi(x)\|^2} \\ &= \sqrt{2} \|z\| \|\psi\| \quad \forall z \in \mathbb{G}, \forall \psi \in L^2(\mathbb{R}, \mathbb{G}) \end{aligned} \tag{3.15}$$

Furthermore one has that:

$$\begin{aligned} \|(f, g)\| &= \left\| \int_{-\infty}^{+\infty} dx \bar{f}(x)g(x) \right\| \leq \int_{-\infty}^{+\infty} dx \|\bar{f}(x)g(x)\| \\ &\leq \sqrt{2} \int_{-\infty}^{+\infty} dx \|\bar{f}(x)\| \|g(x)\| \leq \sqrt{2} \sqrt{\int_{-\infty}^{+\infty} dx \|f(x)\|^2} \\ &\quad \times \sqrt{\int_{-\infty}^{+\infty} dx \|g(x)\|^2} = \sqrt{2} \|f\| \|g\| \quad \forall f, g \in L^2(\mathbb{R}, \mathbb{G}) \end{aligned} \tag{3.16}$$

from which, using the fact that absolute convergence implies convergence, it follows that:

$$(f, g) \in \mathbb{G} \quad \forall f, g \in L^2(\mathbb{R}, \mathbb{G}) \tag{3.17}$$

$(L^2(\mathbb{R}, \mathbb{G}), (\cdot, \cdot), \|\cdot\|)$  is then an hyperbolic Hilbert space.

As to unbounded operators over an hyperbolic Hilbert space  $\mathcal{H}$  let us observe that, as in the analogous case of unbounded operators over a (complex) Hilbert space (Reed and Simon, 1980), they will be usually defined only on a dense linear subspace of  $\mathcal{H}$ .

#### 4. HYPERBOLIC QUANTUM MECHANICS

Hyperbolic numbers emerged in the research of one of the authors (Khrennikov, 2003a, 2003b, 2004) as the underlying number system of a

mathematical theory, “Hyperbolic Quantum Mechanics” formalized by the following axioms:

**Axiom 4.1.** *The pure states of an hyperbolic quantum systems are rays on an hyperbolic Hilbert space  $\mathcal{H}$*

**Axiom 4.2.** *Hyperbolic quantum mechanical observables are linear operators on  $\mathcal{H}$  having real spectrum. The expected value of the hyperbolic observable  $\hat{O}$  in a state  $\psi \in \mathcal{H}$  such that  $(\psi, \psi) \neq 0$  is given by:*

$$E_\psi(O) = \frac{(\psi, O\psi)}{(\psi, \psi)} \tag{4.1}$$

**Axiom 4.3.** *The evolution of a pure state  $\psi_0 \in \mathcal{H}$  is described by the hyperbolic analogue of Schrödinger’s equation:*

$$j \frac{d\psi(t)}{dt} = H\psi(t), \psi(0) = \psi_0 \tag{4.2}$$

### 5. ABOUT VON NEUMANN UNIQUENESS THEOREM

Let us leave aside for a moment Hyperbolic Quantum Mechanics and let us analyze the status of Von Neumann Uniqueness Theorem in ordinary (complex) Quantum Mechanics.

Given an Hilbert space  $\mathcal{H}$ , a dense linear subspace  $\mathcal{D}$  of  $\mathcal{H}$  and two linear operators  $\hat{Q}, \hat{P}$  over  $\mathcal{H}$  we will say that:

*Definition 5.1.* ( $\hat{Q}$  and  $\hat{P}$  are a Representation of the Canonical Commutation Relation Over  $\mathcal{D}$ )

- $\mathcal{D} \subseteq D(\hat{Q}) \cap D(\hat{P}), \hat{Q}\mathcal{D} \subseteq \mathcal{D}, \hat{P}\mathcal{D} \subseteq \mathcal{D}$
- $([\hat{Q}, \hat{P}])[\psi] := \hat{Q}[\psi]\hat{P}[\psi] - \hat{P}[\psi]\hat{Q}[\psi] = i\hat{I}[\psi] \quad \forall \psi \in \mathcal{D}$

where  $\hat{I}$  is the identity operator over  $\mathcal{H}$ .

Introduced the following operators on  $L^2(\mathbb{R}, \mathbb{C})$ :

$$(\hat{Q}_q\psi)(q) := q\psi(q) \tag{5.1}$$

$$(\hat{P}_q\psi)(q) := -i \frac{d\psi(q)}{dq} \tag{5.2}$$

(where  $i$  is the usual complex imaginary unit such that  $i^2 = -1$ ) defined as the closures of their restriction to the initial domain  $S(\mathbb{R}, \mathbb{C})$ , it may be easily verified

that  $\hat{Q}_q$  and  $\hat{P}_q$  are a representation of the *Canonical Commutation Relation* called the *Schrödinger representation*.

Von Neumann Uniqueness Theorem is often expressed in the Physics' literature as the following:

**Conjecture 5.1.** (Naive Version of Von Neumann Uniqueness Theorem)

HP:

$\hat{Q}$  and  $\hat{P}$  are a representation of the *Canonical Commutation Relation* over the Hilbert space  $\mathcal{H}$

TH:

$$\exists \hat{U} : \mathcal{H} \mapsto L^2(\mathbb{R}, \mathbb{C}) \text{ unitary} : \hat{Q} = \hat{U}^{-1} \hat{Q}_q \hat{U} \text{ and } \hat{P} = \hat{U}^{-1} \hat{P}_q \hat{U}$$

A mathematically more rigorous investigation allows anyway to infer that (Thirring, 1981):

**Theorem 5.1.**

Conjecture 1 is false

**Proof:** Let us consider the following Hilbert space  $(l_2(\mathbb{C}), (\cdot, \cdot))$ :

$$l_2(\mathbb{C}) := \{ \{x_n\}_{n=1}^\infty, x_n \in \mathbb{C} \forall n : \sum_{n=1}^\infty |x_n|^2 < \infty \} \tag{5.3}$$

$$(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) := \sum_{n=1}^\infty \bar{x}_n y_n \tag{5.4}$$

its dense linear subspace:

$$\mathcal{D} := \{ \{x_n\}_{n=1}^\infty \in l_2(\mathbb{C}) : \sum_{n=1}^\infty x_n = 0, \text{ with only finitely many } x_n \neq 0 \} \tag{5.5}$$

Given the infinite-dimensional matrices:

$$\hat{Q} := \text{diagonal } (\mathbb{N}) \tag{5.6}$$

$$\hat{P} := -i \begin{pmatrix} 0 & -1 & -\frac{1}{2} & -\frac{1}{3} & \dots \\ 1 & 0 & -1 & -\frac{1}{2} & \dots \\ \frac{1}{2} & 1 & 0 & -1 & \dots \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.7}$$

one has that  $\hat{Q}$  and  $\hat{P}$  are a representation over the dense linear subspace  $\mathcal{D}$  of the *Canonical Commutation Relation* unitarily inequivalent to the Schrödinger representation.

For other contra-examples see Summers, 2001 and references therein □

Theorem 5. 1 has led most of the Mathematical Physics' community to consider representations not of the *Canonical Commutation Relation* but of the following *Weyl relation*:

$$\hat{V}_1(t)\hat{V}_2(s) = \exp(its)\hat{V}_2(s)\hat{V}_1(t) \quad \forall t, s \in \mathbb{R} \tag{5.8}$$

of which the strongly continuous unitary groups  $\{\exp(it\hat{P}_q)\}_{t \in \mathbb{R}}$  and  $\{\exp(is\hat{Q}_q)\}_{s \in \mathbb{R}}$  are indeed a representation, and to call Von Neumann Uniqueness Theorem the following theorem (for whose proof we demand to (Reed and Simon, 1979)):

**Theorem 5.2.** (On the Uniqueness of Representations of Weyl Relation)

HP:

$\{\hat{V}_1(t)\}_{t \in \mathbb{R}}, \{\hat{V}_2(s)\}_{s \in \mathbb{R}}$  one parameter strongly-continuous unitary group on a separable Hilbert space  $\mathcal{H}$  satisfying the *Weyl relation*

TH:

There are closed linear subspaces  $\mathcal{H}_l$  such that:

- $\mathcal{H} = \bigoplus_{l=1}^N \mathcal{H}_l \quad N \in \mathbb{N}_+ \cup \{\infty\}$
- $\hat{U}(t) : \mathcal{H}_l \mapsto \mathcal{H}_l, \hat{V}(s) : \mathcal{H}_l \mapsto \mathcal{H}_l \quad \forall s, t \in \mathbb{R}$
- $\forall l, \exists \hat{T}_l : \mathcal{H}_l \mapsto L^2(\mathbb{R}, \mathbb{C})$  unitary :  $\hat{T}_l \hat{U}(t) \hat{T}_l^{-1} = \exp(it\hat{P}_q)$  and  $\hat{T}_l \hat{V}(s) \hat{T}_l^{-1} = \exp(is\hat{Q}_q)$

It is anyway possible to insist on working with the *Canonical Commutation Relation* provided one adds further hypotheses to the Conjecture 5.1 under which it becomes a theorem.

The first step in this direction is rather trivial, consisting simply in getting rid of the reducibility of representations:

given a dense linear subspace  $\mathcal{D}$  of  $\mathcal{H}$ :

*Definition 5.3.* ( $\hat{Q}$  and  $\hat{P}$  are a Self-adjoint Irreducible Representation of the Canonical Commutation Relation Over  $\mathcal{D}$ )

- $\hat{Q}$  and  $\hat{P}$  are self-adjoint
- $\hat{Q}$  and  $\hat{P}$  are a representation of the *Canonical Commutation Relation* over  $\mathcal{D}$
- $\exists \mathcal{I}$  dense linear subspace of  $\mathcal{D}$  :  $(\exp(it\hat{Q})\mathcal{I} \subseteq \mathcal{I} \forall s \in \mathbb{R})$  or  $(\exp(it\hat{P})\mathcal{I} \subseteq \mathcal{I} \forall t \in \mathbb{R})$

Among the many possibilities one is the following (Summers, 2001):

**Theorem 5.3.** (Von Neumann Uniqueness Theorem (in Weakened Dixmier's Form))

HP:

$\hat{Q}$  and  $\hat{P}$  are a self-adjoint irreducible representation of the *Canonical Commutation Relation* over a dense linear subspace  $\mathcal{D}$  of an Hilbert space  $\mathcal{H}$  such that  $\hat{Q}$  and  $\hat{P}$  are closed and the restriction of  $\hat{Q}^2 + \hat{P}^2$  to  $\mathcal{D}$  is essentially self-adjoint

TH:

$$\exists \hat{U} : \mathcal{H} \mapsto L^2(\mathbb{R}, \mathbb{C}) \text{ unitary} : \hat{Q} = \hat{U}^{-1} \hat{Q}_q \hat{U} \text{ and } \hat{P} = \hat{U}^{-1} \hat{P}_p \hat{U}$$

The *Schrödinger representation* of the *Canonical Commutation Relation* is also called the *position representation*.

Let us now introduce the following operators:

$$(\hat{Q}_p \psi)(p) := +i \frac{d\psi(p)}{dp} \tag{5.9}$$

$$(\hat{P}_p \psi)(p) := p\psi(p) \tag{5.10}$$

defined as the closures of their restriction to the initial domain  $\mathcal{S}(\mathbb{R}, \mathbb{C})$ .

It may be easily verified that  $\hat{Q}_p$  and  $\hat{P}_p$  are an irreducible self-adjoint representation of the *Canonical Commutation Relation*, called the *momentum representation*, over a dense linear subspace of  $L^2(\mathbb{R}, \mathbb{C})$  over which  $\hat{Q}_p^2 + \hat{P}_p^2$  is essentially self-adjoint.

Applying Theorem 5.3 it follows that:

**Corollary 5.1.** (On the Unitarily Equivalence of the Position and Momentum Representations)

$$\exists \hat{U} : \mathcal{H} \mapsto L^2(\mathbb{R}, \mathbb{C}) \text{ unitary} : \hat{Q}_p = \hat{U}^{-1} \hat{Q}_q \hat{U} \text{ and } \hat{P}_p = \hat{U}^{-1} \hat{P}_q \hat{U}$$

Indeed the unitary of Corollary 5.1 is nothing but the usual (complex) Fourier transform.

## 6. POSITION AND MOMENTUM REPRESENTATIONS OF THE HYPERBOLIC CANONICAL COMMUTATION RELATION

Given the *Hyperbolic Canonical Commutation Relation*:

$$[\hat{q}, \hat{p}] = j\hat{I} \quad (6.1)$$

let us consider its *position representation* in  $L^2(\mathbb{R}, \mathbb{G})$ :

$$(\hat{q}_q \psi)(q) := q\psi(q) \quad (6.2)$$

$$(\hat{p}_q \psi)(q) := -j \frac{d\psi(q)}{dq} \quad (6.3)$$

and its *momentum representation* in  $L^2(\mathbb{R}, \mathbb{G})$ :

$$(\hat{q}_p \psi)(p) := +j \frac{d\psi(p)}{dp} \quad (6.4)$$

$$(\hat{p}_p \psi)(p) := p\psi(p) \quad (6.5)$$

where all the operators are defined as the closures of their restriction on the initial domain  $\mathcal{S}(\mathbb{R}, \mathbb{G})$ .

We will prove the following:

**Theorem 6.1.** (On the Unitarily Inequivalence of the Hyperbolic Position and Momentum Representations)

$\exists \hat{U} : L^2(\mathbb{R}, \mathbb{G}) \mapsto L^2(\mathbb{R}, \mathbb{G})$  unitary :

$$\hat{q}_p = \hat{U}^{-1} \hat{q}_q \hat{U}$$

$$\hat{p}_p = \hat{U}^{-1} \hat{p}_q \hat{U}$$

**Proof:** Owing to Theorem B.1 we know that the required  $\hat{U}$  is not the Fourier transform as instead occurs in Quantum Mechanics.

Let us now follow for a moment the non-rigorous Dirac bra-ket formalism.

Starting with:

$$\langle q|\hat{q}|\alpha\rangle = q\langle q|\alpha\rangle \tag{6.6}$$

$$\langle q|\hat{p}|\alpha\rangle = -j\frac{d}{dq}\langle q|\alpha\rangle \tag{6.7}$$

one has in particular that:

$$\langle q|\hat{p}|p\rangle = -j\frac{d}{dq}\langle q|p\rangle \tag{6.8}$$

and hence:

$$\left[ j\frac{d}{dq} + p \right] \langle q|p\rangle = 0 \tag{6.9}$$

from which it follows that:

$$\langle q|p\rangle = c \exp(-j pq) \quad c \in \mathbb{G} \tag{6.10}$$

But then one has that:

$$\begin{aligned} \langle p|\hat{q}|\alpha\rangle &= \int_{-\infty}^{+\infty} dq \langle p|q\rangle \langle q|\hat{q}|\alpha\rangle = \bar{c} \int_{-\infty}^{+\infty} dq \exp(j pq) \langle q|\hat{q}|\alpha\rangle \\ &= \bar{c} \int_{-\infty}^{+\infty} dq \exp(j pq) q \langle q|\alpha\rangle = \frac{\bar{c}}{j} \frac{d}{dp} \int_{-\infty}^{+\infty} dq \exp(j pq) \langle q|\alpha\rangle \\ &= j \frac{d}{dp} \int_{-\infty}^{+\infty} dq \langle p|q\rangle \langle q|\alpha\rangle = +j \frac{d}{dp} \langle p|\alpha\rangle = (\hat{q}_p \psi_{|\alpha\rangle})(p) \end{aligned} \tag{6.11}$$

where we have used Eq. (6.10) and the completeness condition for position autokets:

$$\int_{-\infty}^{+\infty} dq |q\rangle \langle q| = \hat{I} \tag{6.12}$$

Equation (6.11) implies that:

$$\hat{q}_p[f] = (\mathcal{F}\hat{q}_q\mathcal{F}^{-1})[f] \quad \forall f \in N[\mathcal{F}] \tag{6.13}$$

$$\hat{p}_p[f] = (\mathcal{F}\hat{p}_q\mathcal{F}^{-1})[f] \quad \forall f \in N[\mathcal{F}] \tag{6.14}$$

where:

$$N[\mathcal{F}] := \{ f \in D(\mathcal{F}) : \mathcal{F}[f] \in \mathcal{S}(\mathbb{R}, \mathbb{G}) \} \tag{6.15}$$

So, assuming ad absurdum the existence of a unitary  $\hat{U} : L^2(\mathbb{R}, \mathbb{G}) \mapsto L^2(\mathbb{R}, \mathbb{G})$ :

$$\hat{q}_p = \hat{U}^{-1} \hat{q}_q \hat{U} \tag{6.16}$$

$$\hat{p}_p = \hat{U}^{-1} \hat{p}_q \hat{U} \tag{6.17}$$

one should have that  $\hat{U}|_{N[\mathcal{F}]} = \mathcal{F}^{-1}$ . □

The same formulation of a conjecture claiming the existence an an analogous of Theorem 5.3 for operators on an hyperbolic Hilbert space would be an highly not trivial task owing to the peculiarities of self-adjoint operators on such a space discussed in the Appendix C.

Theorem 6.1, anyway, automatically implies that such a conjecture would be false, i.e. that Von Neumann Uniqueness Theorem doesn't hold in Hyperbolic Quantum Mechanics.

In fact, if an hyperbolic quantum mechanical analogous of Theorem 5.3 existed, it would imply the violation of Theorem 6.1.

**APPENDIX A: HYPERBOLIC FUNCTIONS AT RAPID DECREASE AND HYPERBOLIC TEMPERED DISTRIBUTIONS**

*Definition A.1.* (Hyperbolic Functions of Rapid Decrease)

$$\begin{aligned} \mathcal{S}(\mathbb{R}, \mathbb{G}) &:= \left\{ f : \mathbb{R} \rightarrow \mathbb{G} \text{ infinitely differentiable} : \|f\|_{n,m} : \right. \\ &= \left. \sup_{x \in \mathbb{R}} \left\| x^n \frac{d^m}{dx^m} f(x) \right\| < \infty \forall n, m \in \mathbb{N}_+ \right\} \end{aligned}$$

Let us endow  $\mathcal{S}(\mathbb{R}, \mathbb{G})$  with the natural topology induced by the seminorms  $\| \cdot \|_{n,m}$ .

*Definition A.2.* (Space of Hyperbolic Tempered Distributions)

$$\mathcal{S}'(\mathbb{R}, \mathbb{G}) := \text{topological-dual}[\mathcal{S}(\mathbb{R}, \mathbb{G})]$$

In particular let us introduce the following:

*Definition A.3.* (Hyperbolic Dirac Delta)

$$\delta \in \mathcal{S}'(\mathbb{R}, \mathbb{G}) : \delta[f] := f(0)$$

Given an hyperbolic tempered distribution  $\lambda \in \mathcal{S}'(\mathbb{R}, \mathbb{G})$ , a family of functions  $f_\alpha : \mathbb{R} \rightarrow \mathbb{G}$  for every  $\alpha \in I := [a, b]$  with  $a, b \in [0, +\infty]$ , a measure  $\mu$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and a number  $\bar{\alpha} \in I$ :

*Definition A.4.* ( $f_\alpha$  Is a Limit-Representation of  $\lambda$  with Respect to  $\mu$  for  $\alpha \rightarrow \bar{\alpha}$  (REP $_\mu - \lim_{\alpha \rightarrow \bar{\alpha}} f_\alpha = \lambda$ )



$$\lim_{\alpha \rightarrow \bar{\alpha}} \int d\mu(x) f_\alpha(x) \phi(x) = \lambda[\phi] \quad \forall \phi \in \mathcal{S}(\mathbb{R}, \mathbb{G}) \tag{A.1}$$

Let us now consider the family of functions:

$$f_\alpha(x) := \int_{-\alpha}^{\alpha} dp \exp(jpx) = \frac{2 \sinh(\alpha x)}{x} \tag{A.2}$$

One has that:

**Theorem A.1.**

$$REP_{\mu_{\text{Lebesgue}}} - \lim_{\alpha \rightarrow +\infty} f_\alpha \neq \delta$$

**Proof:** Let us consider the test function  $\phi(x) := \exp(-x^2)$ .

Since:

$$\int_{-\alpha}^{\alpha} dp \exp(jpx - x^2) \tag{A.3}$$

doesn't converge to  $\phi(0) = 1$  as  $\alpha \rightarrow \infty$  the thesis follows □

One has that:

**Theorem A.2.** (Embedding Theorem)

$$\mathcal{S}(\mathbb{R}, \mathbb{G}) \subset L^2(\mathbb{R}, \mathbb{G}) \subset \mathcal{S}'(\mathbb{R}, \mathbb{G})$$

**Proof:** Every function  $f \in \mathcal{S}(\mathbb{R}, \mathbb{G})$  can be identified with the functional  $f[\cdot] \in \mathcal{S}'(\mathbb{R}, \mathbb{G})$  defined as:

$$f[g] := \int_{-\infty}^{+\infty} dx g(x) f(x) \tag{A.4}$$

From the other side one has that:

$$(\|f\|_{n,m} < +\infty \quad \forall n, m \in \mathbb{N}_+) \Rightarrow \int_{-\infty}^{+\infty} dx \|f(x)\|^2 \in (-\infty, +\infty) \tag{A.5}$$

and hence  $f \in L^2(\mathbb{R}, \mathbb{G})$  □

**APPENDIX B. THE HYPERBOLIC FOURIER TRANSFORM**

Let us introduce the following:

*Definition B.1.* (Hyperbolic Fourier Transform:)

the functional  $\mathcal{F} : D(\mathcal{F}) \mapsto MAPS(\mathbb{R}, \mathbb{G})$ :

$$D(\mathcal{F}) := \{f \in \mathcal{S}(\mathbb{R}, \mathbb{G}) : \exists \text{finite } \int_{-\infty}^{+\infty} dx \exp(-jpx)f(x)\}$$

$$(\mathcal{F}[f])(p) := \int_{-\infty}^{+\infty} dx \exp(-jpx)f(x)$$

Let us observe that:

**Proposition B.1.**

$$\mathcal{F}[D(\mathcal{F})] \not\subseteq \mathcal{S}(\mathbb{R}, \mathbb{G})$$

**Proof:** Given the function  $f(x) := \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) \in \mathcal{S}(\mathbb{R}, \mathbb{G})$  one has that:

$$\mathcal{F}[f](p) = \exp(\frac{p^2}{2}) \notin \mathcal{S}(\mathbb{R}, \mathbb{G}) \tag{B.1}$$

□

Proposition 5.1 implies that:

**Theorem B.1.** (No Hyperbolic Plancherel Theorem:)

$\mathcal{F}$  doesn't extend uniquely to a unitary  $\hat{U} : L^2(\mathbb{R}, \mathbb{G}) \mapsto L^2(\mathbb{R}, \mathbb{G})$

**APPENDIX C: SELF-ADJOINT OPERATORS ON AN HYPERBOLIC HILBERT SPACE**

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be hyperbolic Banach spaces. Given a bounded linear operator  $T : X \mapsto Y$  let us introduce the following:

*Definition C.1.* (Hyperbolic Banach Space Adjoint of T)

the operator  $T' : Y^* \mapsto X^*$ :

$$(T'l)(x) := l(Tx) \quad l \in Y^*, x \in X \tag{C.1}$$

Let us now consider an hyperbolic Hilbert space  $(\mathcal{H}, (\cdot, \cdot), \|\cdot\|)$ . The hyperbolic Banach space adjoint of a bounded linear operator  $T : \mathcal{H} \mapsto \mathcal{H}$  is then an operator  $T' : \mathcal{H}^* \mapsto \mathcal{H}^*$ .

Let us now consider the map  $C : \mathcal{H} \mapsto \mathcal{H}^*$  which assigns to each  $y \in \mathcal{H}$  the linear functional  $(y, \cdot) \in \mathcal{H}^*$ .

The key difference with respect to the analogous situation on an Hilbert space (Reed and Simon, 1980) is that in our case there is no analogue of Riesz Lemma and hence nobody assures us that  $C$  is surjective.

We are thus led to the following:

*Definition C.2.* (Hyperbolic Hilbert Space Adjoint of  $T$ )  
 the operator  $T^\dagger : D(T^\dagger) \mapsto \mathcal{H}$ :

$$T^\dagger := C^{-1}T'C \tag{C.2}$$

where  $D(T^\dagger)$  is the linear subspace of  $\mathcal{H}$  where  $C^{-1}T'C$  is well-defined.

The notion of hyperbolic Hilbert space adjoint can be extended to unbounded operators in the following way:

given an unbounded linear operator  $T$  defined on a dense subspace  $D(T)$  of an hyperbolic Hilbert space  $\mathcal{H}$ :

*Definition C.3.* (Hyperbolic Hilbert Space Adjoint of  $T$ )  
 the operator  $T^\dagger : D(T^\dagger) \mapsto \mathcal{H}$ :

$$D(T^\dagger) := \{\phi \in \mathcal{H} : (\exists ! \eta_\phi \in \mathcal{H} : (T\psi, \phi) = (\psi, \eta_\phi) \forall \psi \in D(T))\}$$

$$T^\dagger \phi := \eta_\phi$$

*Definition C.4.* ( $T$  is Self-Adjoint)

$$D(T) = D(T^\dagger) \text{ and } T = T^\dagger$$

A standard theorem of Functional Analysis asserts that the spectrum of a self-adjoint operator on an Hilbert space is a subset of the real line (Reed and Simon, 1980).

This is no more true as to self-adjoint operators on a hyperbolic Hilbert space as we will show in the simplest case  $\mathcal{H} = \mathbb{G}^2$ .

Let us introduce at this purpose the following useful bijection  $T : M_n(\mathbb{G}) \mapsto M_n(\mathbb{C})$ :

$$T(\{x_{i,j} + jy_{i,j}\}_{i,j=1}^n) := \{x_{i,j} + iy_{i,j}\}_{i,j=1}^n \tag{C.3}$$

Given:

$$A := \begin{pmatrix} x_{11} + jy_{11} & x_{12} + jy_{12} \\ x_{21} + jy_{21} & x_{22} + jy_{22} \end{pmatrix} \in M_2(\mathbb{G}) \tag{C.4}$$

one has that:

$$A^\dagger = \bar{A}^t = \begin{pmatrix} x_{11} - jy_{11} & x_{21} - jy_{21} \\ x_{12} - jy_{12} & x_{22} - jy_{22} \end{pmatrix} \tag{C.5}$$

Let us introduce the set of self-adjoint matrices:

$$SA_2(\mathbb{G}) := \{A \in M_2(\mathbb{G}) : A^\dagger = A\} \tag{C.6}$$

Clearly one has that:

$$A \in SA_2(\mathbb{G}) \Leftrightarrow y_{11} = 0 \text{ and } y_{22} = 0 \text{ and } x_{12} = x_{21} \text{ and } y_{12} = -y_{21} \tag{C.7}$$

so that the generic matrix  $A \in SA_2(\mathbb{G})$  is of the form:

$$A = \begin{pmatrix} x_{11} & x_{12} + jy_{12} \\ x_{12} - jy_{12} & x_{22} \end{pmatrix} \tag{C.8}$$

Let us compare the eigenvalue equation of A and T(A). For the latter the equation:

$$\det(T(A) - \lambda I) = 0 \tag{C.9}$$

has solution:

$$\lambda = \frac{x_{11} + x_{22} \pm \sqrt{(x_{11} - x_{22})^2 + 4x_{12}^2 + 4y_{12}^2}}{2} \tag{C.10}$$

and since the discriminant  $\Delta := (x_{11} - x_{22})^2 + 4x_{12}^2 + 4y_{12}^2 \geq 0$ :

- if  $\Delta := (x_{11} - x_{22})^2 + 4x_{12}^2 + 4y_{12}^2 > 0$  then T(A) has 2 real eigenvalues:

$$\lambda_1 = \frac{x_{11} + x_{22} + \sqrt{\Delta}}{2} \in \mathbb{R} \tag{C.11}$$

$$\lambda_2 = \frac{x_{11} + x_{22} - \sqrt{\Delta}}{2} \in \mathbb{R} \tag{C.12}$$

- if  $\Delta := (x_{11} - x_{22})^2 + 4x_{12}^2 + 4y_{12}^2 = 0$  then T(A) has 1 real eigenvalue:

$$\lambda = \frac{x_{11} + x_{22}}{2} \in \mathbb{R} \tag{C.13}$$

As to A, instead, the equation:

$$\det(A - \lambda I) = 0 \tag{C.14}$$

has solution:

$$\lambda = \frac{x_{11} + x_{22} \pm \sqrt{(x_{11} - x_{22})^2 + 4x_{12}^2 - 4y_{12}^2}}{2} \tag{C.15}$$

It follows that:

- if  $\Delta := (x_{11} - x_{22})^2 + 4x_{12}^2 - 4y_{12}^2 > 0$  then A has 4 eigenvalues of which only two are reals:

$$\lambda_1 = \frac{x_{11} + x_{22} + \sqrt{\Delta}}{2} \in \mathbb{R} \tag{C.16}$$

$$\lambda_2 = \frac{x_{11} + x_{22} - \sqrt{\Delta}}{2} \in \mathbb{R} \tag{C.17}$$

$$\lambda_3 = \frac{x_{11} + x_{22} + j\sqrt{\Delta}}{2} \notin \mathbb{R} \tag{C.18}$$

$$\lambda_4 = \frac{x_{11} + x_{22} - j\sqrt{\Delta}}{2} \notin \mathbb{R} \tag{C.19}$$

- if  $\Delta := (x_{11} - x_{22})^2 + 4x_{12}^2 - 4y_{12}^2 = 0$  then A has 1 real eigenvalue:

$$\lambda = \frac{x_{11} + x_{22}}{2} \in \mathbb{R} \tag{C.20}$$

- if  $\Delta := (x_{11} - x_{22})^2 + 4x_{12}^2 - 4y_{12}^2 < 0$  then A has no eigenvalues

In particular we have shown that a matrix  $A \in SA_2(\mathbb{G})$  cannot be always diagonalized, a fact that by itself proves that the Spectral Theorem doesn't hold for self-adjoint operators on an hyperbolic Hilbert space.

This fact implies that given a linear operator A on an Hyperbolic Hilbert space:

- if A is bounded, the exponential of A can be defined by power-series:

$$\exp(jtA) := \sum_{n=0}^{\infty} \frac{(jt)^n A^n}{n!} \tag{C.21}$$

- if A is unbounded, not only the exponential of A cannot be defined by power series (as occurs also for an unbounded operator on a (complex) Hilbert space), but one cannot use the functional calculus form of the Spectral theorem; it follows that no definition of  $\exp(jtA)$  can be given in this way.

As a consequence it follows that no analogue exists on an Hyperbolic Hilbert Space of the Stone Theorem that on a (complex) Hilbert space states the existence of a bijection between self-adjoint operators and strongly-continuous unitary groups associating to each self-adjoint operator A the strongly-continuous unitary group  $\{\exp(itA)\}_{t \in \mathbb{R}}$ .

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